

A note on unitizations of generalized effect algebras

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Abstract There is a forgetful functor from the category of generalized effect algebras to the category of effect algebras. We prove that this functor is a right adjoint and that the corresponding left adjoint is the well-known unitization construction by Hedlíková and Pulmannová. Moreover, this adjunction is monadic.

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1 Introduction and preliminaries

1.1 Introduction

In [5], authors proved that every generalized effect algebra can be embedded into an effect algebra. The construction was subsequently studied and applied by several authors, for example in [10], [12], [11]. A generalization of the unitization construction to pseudoeffect algebras was recently introduced and studied in [3]

It is easy to see that this unitization construction is functorial. We prove that this *unitization functor* is left adjoint to the forgetful functor from the generalized effect algebras to effect algebras and that this adjunction is monadic. Thus, the category of effect algebras

is the category of algebras for a monad defined on the category of generalized effect algebras.

We assume working knowledge of basic category theory [8] and theory of effect algebras [1].

Let P be a partial algebra with a nullary operation 0 and a binary partial operation \oplus . Denote the domain of \oplus by \perp . P is called a *generalized effect algebra* iff for all $a, b, c \in P$ the following conditions are satisfied :

- (P1) $a \perp b$ implies $b \perp a$, $a \oplus b = b \oplus a$.
- (P2) $b \perp c$ and $a \perp b \oplus c$ implies $a \perp b$, $a \oplus b \perp c$,
 $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
- (P3) $a \perp 0$ and $a \oplus 0 = a$.
- (P4) $a \oplus b = a \oplus c$ implies $b = c$.
- (P5) $a \oplus b = 0$ implies $a = 0$.

In a generalized effect algebra P , we denote $a \leq b$ iff $a \oplus c = b$ for some $c \in P$. It is easy to see that \leq is a partial order and that 0 is the least element of the poset (P, \leq) . We denote $c = a \ominus b$ iff $a = b \oplus c$. Owing to cancellativity, \ominus is a well-defined partial operation with domain \geq .

Let P_1, P_2 be generalized effect algebras. A map $f : P_1 \rightarrow P_2$ is called a *morphism of generalized effect algebras* if and only if it satisfies the following conditions.

- $f(0) = 0$.
- If $a \perp b$, then $f(a) \perp f(b)$ and $f(a \oplus b) = f(a) \oplus f(b)$.

A morphism is $f : P_1 \rightarrow P_2$ is *full* if $f(a) \perp f(b)$ implies that there are $a_1, b_1 \in P_1$ such that $a_1 \perp b_1$, $f(a) = f(a_1)$ and $f(b) = f(b_1)$.

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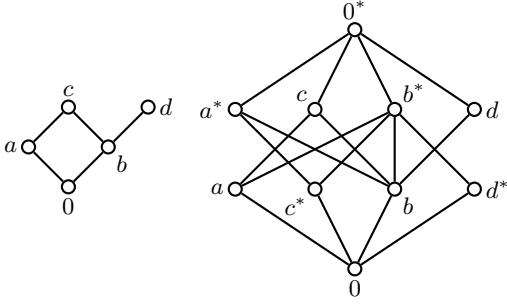


Fig. 1 A generalized effect algebra and its unitization

1.3 Effect algebras

An *effect algebra* is an generalized effect algebra bounded above. Unwinding this definition, we observe that an effect algebra is partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation \oplus and two nullary operations $0, 1$ such that the reduct $(E; \oplus, 0)$ is a generalized effect algebra and 1 is the greatest element of E .

Effect algebras were introduced by Foulis and Bennett in their paper [2]. See also [7] and [4] for equivalent definitions, introduced independently.

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \rightarrow E_2$ is called a *morphism of effect algebras* if and only if it is a morphism of generalized effect algebras satisfying the condition $\phi(1) = 1$. A morphism $\phi : E_1 \rightarrow E_2$ is a *full morphism* if and only if $\phi(a) \perp \phi(b)$ implies that there are $x, y \in E_1$ such that $\phi(x) = a$, $\phi(y) = b$ and $x \perp y$. A full, bijective morphism is an *isomorphism*.

2 The unitization functor

The category of generalized effect algebras is denoted by **GEA**, the category of effect algebras is denoted by **EA**, $U : \mathbf{EA} \rightarrow \mathbf{GEA}$ is the evident forgetful functor.

Let us define a functor $F : \mathbf{GEA} \rightarrow \mathbf{EA}$, called *unitization*.

For a generalized effect algebra P , $F(P)$ is a partial algebra with an underlying set $P \dot{\cup} P^*$, where $P \cap P^* = \emptyset$ and $a \mapsto a^*$ is a bijection from P to P^* , equipped with a partial operation given as follows: for all $a, b \in P$,

- $a \perp_{F(P)} b$ iff $a \perp_P b$ and then $a \oplus_{F(P)} b = a \oplus_P b$,
- $a \perp_{F(P)} b^*$ iff $a \leq b$ and then $a \oplus_{F(P)} b^* = (b \ominus_P a)^*$,
- $a^* \perp_{F(P)} b$ iff $a \geq b$ and then $a^* \oplus_{F(P)} b = (a \ominus_P b)^*$,
- $a^* \not\perp_{F(P)} b^*$.

This construction was introduced by Hedlíková and Pulmannová in [5]. They proved that $(F(P), \oplus, 0, 0^*)$ is always an effect algebra. The basic idea of the construction predates effect algebras, see [6], [9].

Example 1 Let P be the poset on the left-hand side of Figure 1. There is a unique \oplus partial operation on

P , making $(P, \oplus, 0)$ into a generalized effect algebra: $a \oplus b = b \oplus a = c$, $b \oplus b = d$ and $0 \oplus x = x \oplus 0 = x$, for all $x \in P$.

The Hasse diagram of the effect algebra $F(P)$ appears on the right-hand side of the picture.

Example 2 Let us consider generalized effect algebra (in fact, a total monoid) $(\mathbb{N}, +, 0)$, where $+$ is the ordinary addition of natural numbers. Then $F(\mathbb{N})$ is a totally ordered MV-algebra, also known under the name *Chang's MV-algebra*.

For a morphism of generalized effect algebras $f : P \rightarrow Q$, then $F(f) : F(P) \rightarrow F(Q)$ is given by $F(f)(a) = a$, $F(f)(a^*) = (f(a))^*$. It is easy to check that $F(f)$ is a morphism in the category **EA** and that $F : \mathbf{GEA} \rightarrow \mathbf{EA}$ is a functor.

Theorem 1 F is left adjoint to U .

Proof Let us define the unit η . For every generalized effect algebra P , the component $\eta_P : P \rightarrow UF(P)$ is the embedding $\eta_P(x) = x$. This is obviously a natural transformation $1_{\mathbf{GEA}} \rightarrow UF$.

Let E be an effect algebra; to define the component of the counit ϵ at E , we need to take a closer look at $FU(E)$. Let us prove that $w : FU(E) \rightarrow E \times \{0, 1\}$ given by $w(a) = (a, 0)$ and $w(a^*) = (a', 1)$, where $a \in E$, is an isomorphism of effect algebras. Indeed, suppose that $x, y \in FU(E)$ and that $x \perp y$. The only nontrivial case we have to check is when there are $a, b \in E$ such that $x = a$, $y = b^*$ and $a \leq b$; in this case $x \oplus y = (b \ominus_E a)^*$ and

$$\begin{aligned} w(x \oplus y) &= w((b \ominus_E a)^*) = ((b \ominus a)', 1) \\ &= (b' \oplus a, 1) = (a, 0) \oplus (b', 1) = w(a) \oplus w(b^*) = w(x) \oplus w(y). \end{aligned}$$

The morphism w is easily seen to be full and bijective, hence an isomorphism.

We may now define $\epsilon_E : FU(E) \rightarrow E$ as the composition of w with the canonical projection $p : E \times \{0, 1\} \rightarrow E$. Explicitly, $\epsilon_E(x) = x$ for $x \in E$ and $\epsilon_E(x^*) = x'$ for $x^* \in E^*$. The commutativity of the naturality square of ϵ also clear.

Let us check the triangle identities. We need to prove that, in the categories of endofunctors of **GEA** and **EA**, respectively, the triangles

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FUF \\ & \searrow 1_F & \downarrow \epsilon_F \\ & & F \end{array} \quad \begin{array}{ccc} U & \xrightarrow{\eta U} & UFU \\ & \searrow 1_U & \downarrow U\epsilon \\ & & U \end{array}$$

commute.

To observe the commutativity of the left triangle, let P be a generalized effect algebra. If $x \in P$, then

$F(\eta_P)(x) = x$ and $\epsilon_{F(P)}(x) = x$. If $x^* \in P$, then $F(\eta_P(x^*)) = (\eta_P(x))^* = x^*$ and $\epsilon_{F(P)}(x^*) = x^*$.

To observe the commutativity of the right triangle, let E be an effect algebra and let $x \in U(E)$. Then $\eta_{U(E)}(x) = x$ and $U(\epsilon_{FU}(x)) = x$.

Let us consider the real interval $[0, 1]_{\mathbb{R}}$, equipped with the usual addition of real numbers restricted to $[0, 1]_{\mathbb{R}}$, meaning that $a \oplus b$ is defined if and only if $a + b \leq 1$ and then $a \oplus b := a + b$. A morphism of generalized effect algebras $P \rightarrow [0, 1]_{\mathbb{R}}$ is called an *additive map on P* .

For an effect algebra E , a *state on E* is an additive map $E \rightarrow [0, 1]_{\mathbb{R}}$ preserving the unit, so a state is a morphism in **EA**.

Corollary 1 *Every additive map on P uniquely extends to a state on $F(P)$.*

Proof If s is an additive map, then there is a unique $\bar{s} : F(P) \rightarrow [0, 1]_{\mathbb{R}}$ such that the diagram

$$\begin{array}{ccc} UF(P) & \xrightarrow{U(\bar{s})} & U([0, 1]_{\mathbb{R}}) \\ \uparrow \eta_P & \nearrow f & \\ P & & \end{array}$$

commutes.

Every state f on an effect algebra must satisfy $f(x') = 1 - f(x)$. Therefore, if s is an additive map on P , then the state \bar{s} on $F(P)$ corresponding to s via the bijection established in 1 is necessarily given by

$$\bar{s}(x) = \begin{cases} s(x) & x \in P \\ 1 - s(x) & x \in P^* \end{cases}$$

In a very similar way, one can prove the following:

Corollary 2 *There is a natural one-to-one correspondence between ideals of P and morphisms $F(P) \rightarrow 2^2$, where 2^2 is the Boolean algebra with two atoms.*

Lemma 1 *The forgetful functor $U : \mathbf{EA} \rightarrow \mathbf{GEA}$ creates coequalizers.*

Proof Let $f, g : A \rightarrow B$ be a pair of morphisms in **EA**, let $h : B \rightarrow Z$ be a coequalizer of f, g in **GEA**. We need to prove that $h(1)$ is the top element of Z . Consider the diagram

$$\begin{array}{ccccc} U(A) & \xrightleftharpoons[U(g)]{U(f)} & U(B) & \xrightarrow{h} & Z \\ & & \searrow t & \downarrow j & \uparrow u \\ & & & [0, h(1)]_Z & \end{array}$$

where t is given by $t(x) = h(x)$ and j is the inclusion into Z , so that $j \circ t = h$. Since $t \circ f = t \circ g$, there is a unique $u : Z \rightarrow [0, h(1)]_Z$ such that $u \circ h = t$. This gives us $j \circ u \circ h = h$. Since h is an epimorphism, $j \circ u = id_Z$ and now we see that for every $x \in Z$, $x = j(u(x)) \leq h(1)$, because the range of j is bounded above by $h(1)$.

It remains to prove that h is a coequalizer in **EA**. If E is a generalized effect algebra bounded above and s is a top-preserving morphism such that $s \circ f = s \circ g$, then there is a unique **GEA**-morphism u such that $u \circ h = s$. However, since both h and s preserve the top element, u must be top-preserving as well.

Recall [8], that every adjunction

$$\mathcal{C} \xrightleftharpoons[U]{F} \mathcal{D}$$

gives rise to a monad $(UF, \eta, U\epsilon F)$ on \mathcal{C} . An adjunction is *monadic* if U is equivalent to the forgetful functor coming from the category of algebras \mathcal{C}^{UF} for the monad (UF, η, ϵ) and the comparison gives us then an isomorphism $\mathcal{D} \simeq \mathcal{C}^{UF}$.

Theorem 2 *The adjunction (F, U, η, ϵ) is monadic.*

Proof By Beck's theorem [8], an adjunction is monadic if and only if U creates absolute coequalizers. By Lemma 1, U creates all coequalizers.

Corollary 3 $\mathbf{EA} \simeq \mathbf{GEA}^{UF}$.

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References

1. Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Dordrecht and Ister Science, Bratislava (2000)
2. Foulis, D., Bennett, M.: Effect algebras and unsharp quantum logics. *Found. Phys.* **24**, 1325–1346 (1994)
3. Foulis, D.J., Pulmannová, S.: Unitizing a generalized pseudo effect algebra. *Order* (2014). (to appear)
4. Giuntini, R., Greuling, H.: Toward a formal language for unsharp properties. *Found. Phys.* **19**, 931–945 (1989)
5. Hedlíková, J., Pulmannová, S.: Generalized difference posets and orthoalgebras. *Acta Math. Univ. Comenianae* **65**(2), 247–279 (1996)
6. Janowitz, M.: A note on generalized orthomodular lattices. *J. Natur. Sci. Math* **8**, 89–94 (1968)
7. Kôpka, F., Chovanec, F.: D-posets. *Math. Slovaca* **44**, 21–34 (1994)
8. MacLane, S.: Categories for the Working Mathematician. No. 5 in Graduate Texts in Mathematics. Springer-Verlag (1971)

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9. Mayet-Ippolito, A.: Generalized orthomodular posets. *Demonstratio Math* **24**, 263–274 (1991)
 10. Paseka, J., Riečanová, Z.: Isomorphism theorems on generalized effect algebras based on atoms. *Information Sciences* **179**(5), 521–528 (2009)
 11. Riečanová, Z.: Effect algebraic extensions of generalized effect algebras and two-valued states. *Fuzzy Sets and Systems* **159**(9), 1116–1122 (2008)
 12. Riečanová, Z., Marinová, I.: Generalized homogeneous, prelattice and MV-effect algebras. *Kybernetika* **41**(2), 129–142 (2005)

